International Journal of Theoretical Physics, Vol. 27, No. 6, 1988

Monotone-Iterative Method for Solving the Periodic Problem for Systems of Impulsive Differential Equations

S. G. Hristova[1] and D. D. Bainov[1]

Received November 19, 1987

A modification of the method of monotone operators is applied to the approximate solution of the periodic problem for a nonlinear system of impulsive differential equations.

1. INTRODUCTION

While in recent years the mathematical theory of impulsive systems has undergone intensive research, many of its aspects remain undeveloped, due largely to difficulties related to phenomena of "beating," involving, e.g., a loss of autonomy or a merging of solutions.

The study of the mathematical theory of impulsive systems was initiated by Mil'man and Myshkis (1960).

In the present paper we give a modification of the monotone-iterative method of Lakshmikantham (Bernfield and Lakshmikantham, 1982; Deimling and Lakshmikantham, 1980; Du and Lakshmikantham, 1982; Lakshmikantham and Leela, 1984, Lakshmikantham *et al.*, 1981; Lakshmikantham and Vatsala, 1981; Vatsala, 1983) for the periodic problem for nonlinear systems of impulsive differential equations. Nonlinear periodic roblems for impulsive systems also have been investigated by other methods (Perestyuk and Shovkoplyas, 1973, Samoilenko and Perestyuk, 1982; Hristova and Bainov, 1986).

[1]University of Plovdiv.

0020-7748/88/0600-0757S06.00/0

2. STATEMENT OF THE PROBLEM

Consider the periodic problem for the impulsive system

$$\dot{x} = f(t, x) \qquad \text{for} \quad t \neq t_i$$

$$\Delta x|_{t=t_i} = I_i(x(t_i)) \tag{1}$$

$$x(0) = x(T)$$

where $x \in R^n, f:[0, T] \times R^n \to R^n$, $I_i: R^n \to R^n$ $(i = \overline{1, p})$, $t_i \in (0, T)$ $(i = \overline{1, p})$ are fixed points such that $t_{i+1} > t_i$ $(i = 1, p-1)$, $\Delta x|_{t=t_i} = x(t_i + 0) - x(t_i - 0)$.

We shall say that the function $v:[0, T] \to R^n$ belongs to the set Ω if:

1. The function $v(t)$ is piecewise continuous with points of discontinuity of the first type at the points t_i, $i = \overline{1, p}$.
2. $v(t_i) = v(t_i - 0)$.
3. The function $v(t)$ is continuously differentiable for $t \neq t_i$, $i = \overline{1, p}$.
4. There exists the derivative $\dot{v}(t_i) = \dot{v}(t_i - 0)$ $(i = \overline{1, p})$.

Define the set

$$D(v, w) = \{u \in \Omega : v(t) \leq u(t) \leq w(t) \quad \text{for} \quad t \in [0, T]\}$$

where $v, w \in \Omega$.

Definition 1. The function $v(t) \in \Omega$ is called a lower (upper) solution of periodic problem (1) if

$$\dot{v}(t) \leq (\geq) f(t, v) \qquad \text{for } t \neq t_i$$

$$\Delta v|_{t=t_i} \leq (\geq) I_i(v(t_i))$$

$$v(0) \leq (\geq) v(T)$$

Let $v_0, w_0 \in \Omega$ be respectively, lower and upper solutions of the periodic problem (1).

Definition 2. The function $v \in D(v_0, w_0)$ is called a minimal (maximal) solution of periodic problem (1) in $D(v_0, w_0)$ if it is a solution of (1) and for any other solution $u \in D(u_0, w_0)$ of problem (1) the inequality $v(t) \leq (\geq) u(t)$ holds for $t \in [0, T]$.

Let $u, v \in R^n$, $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$. We shall say that $u \leq v$, if $u_i \leq v_i$ for $i = \overline{1, n}$.

Let $n \geq 2$. To every integer $j \in \overline{1, n}$ we put in correspondence two positive integers p_j and q_j such that $p_j + q_j = n - 1$. Then each element $x \in R^n$ $(n \geq 2)$

can be represented in the form

$$x=(x_j,[x]_{p_j},[x]_{q_j})=\begin{cases}(x_1,\ldots,x_j,\ldots,x_{p_j+1},\underbrace{x_{p_j+2},\ldots,x_n}_{q_d} & \text{for} \quad p_j>j,\\ (x_1,\ldots,\underbrace{x_{p_j},x_{p_j+1},\ldots,x_j,\ldots,x_n)}_{q_j+1} & \text{for} \quad p_j\le j.\end{cases}$$

In terms of the notations introduced, problem (1) for $n>2$ can be written in the form

$$\begin{aligned} \dot{x}_j&=f_j(t,x_j,[x]_{p_j},[x]_{q_j}) \qquad \text{for} \quad t\neq t_i\\ \Delta x_j|_{t=t_i}&=I_{ij}(x_j(t_i),[x(t_i)]_{p_j},[x(t_i)]_{q_j}) \\ x_j(0)&=x_j(T), \qquad j=\overline{1,n}\end{aligned} \tag{2}$$

Definition 3. The functions w, $v\in\Omega$ are called a couple of upper and lower quasisolutions of periodic problem (2) for $n\ge 2$ if

$$\begin{aligned} \dot{v}_j&\le f_j(t,v_j,[v]_{p_j},[w]_{q_j}) \qquad \text{for} \quad t\neq t_i\\ \Delta v_j|_{t=t_i}&\le I_{ij}(v_j(t_i),[v(t_i)]_{p_j},[w(t_i)]_{q_j}) \\ v_j(0)&\le v_j(T), \qquad j=\overline{1,n}\end{aligned} \tag{3}$$

and

$$\begin{aligned} \dot{w}_j&\ge f_j(t,w_j,[w]_{p_j},[v]_{q_j}) \qquad \text{for} \quad t\neq t_i\\ \Delta w_j|_{t=t_i}&\ge I_{ij}(w_j(t_i),[w(t_i)]_{p_j},[v(t_i)]_{q_j}) \\ w_j(0)&\ge w_j(T), \qquad j=\overline{1,n}\end{aligned} \tag{4}$$

Definition 4. The functions v, $w\in\Omega$ are called a couple of quasisolutions of the periodic problem (2) for $n\ge 2$ if the relations (3) and (4) are equalities.

Let v_0, $w_0\in\Omega$ be a couple of lower and upper quasisolutions of problem (2).

Definition 5. The functions v, $w\in\Omega$ are called a couple of minimal and maximal quasisolutions of periodic problem (2) in $D(v_0, w_0)$ if they are a couple of quasisolutions of (2) and for any couple of quasisolutions u_1, $u_2\in\Omega$ of problem (2) the inequalities $v(t)\le u_1(t)\le w(t)$ and $v(t)\le u_2(t)\le w(t)$ hold for $t\in[0,T]$.

Definition 6. The function $f:[0,T]\times R^n\to R^n$ $(n\ge 2)$ is called mixed quasimonotone if for $j=\overline{1,n}$ the function $f_j(t,u_j,[u]_{p_j},[u]_{q_j})$ is monotone nondecreasing with respect to $[u]_{p_j}$ and monotone nonincreasing with respect to $[u]_{q_j}$.

3. MAIN RESULTS

We shall consider separately the cases when problem (1) is a periodic problem for a scalar impulsive equation and when problem (1) is a periodic problem for a system of n $(n \geq 2)$ impulsive equations.

Case I. Let $n = 1$.

Lemma 1. Let the function $m : [0, T] \to R$, $m \in \Omega$, satisfy the inequalities

$$\dot{m}(t) \leq -Mm(t) \qquad \text{for} \quad t \neq t_i$$

$$\Delta m\big|_{t=t_i} \leq -L_i m(t_i)$$

$$m(0) \leq m(T)$$

where $M > 0$, $0 < L_i < 1$ $(i = \overline{1, p})$.

Then, for $t \in [0, T]$ the inequality $m(t) \leq 0$ holds.

Proof. Suppose that this is not true, i.e., there exists a point τ such that $m(\tau) > 0$. Introduce the notation

$$\varepsilon = \sup_{t \in [0,T]} m(t)$$

In virtue of the assumption, the inequality $\varepsilon > 0$ holds.

Consider the following three cases:

Case 1. Let a point $\xi \in [0, T]$ exist, $\xi \neq t_i$ $(i = \overline{1, p})$, such that $m(\xi) = \varepsilon$. If $\xi \in (0, T]$, then the following inequalities hold:

$$m(\xi - h) \leq m(\xi)$$

$$0 \leq \dot{m}(\xi) \leq -Mm(\xi) = -M\varepsilon < 0$$

The contradiction obtained shows that in this case the assumption is not true.

If $\xi = 0$, then the inequality $m(T) \geq m(0) = \varepsilon$ holds. Hence $m(T) = \varepsilon$. Then the inequalities $\dot{m}(T) \geq 0$ and $0 \leq \dot{m}(T) \leq -M\varepsilon < 0$ hold.

The contradiction obtained shows that in this case the assumption is not true.

Case 2. Let a positive integer $\kappa \in \overline{1, p}$ exist such that $m(t_k) > m(t)$ for $t \in [0, T]$. Then there exists a point $\tau \in (0, T)$, $t_{k-1} < \tau < t_k$, such that $m(\tau) > 0$ and $\dot{m}(\tau) \geq 0$. Moreover, the inequalities $0 \leq \dot{m}(\tau) \leq -Mm(\tau) < 0$ hold.

The contradiction obtained shows that in this case the assumption is not true.

Case 3. Let an integer $k \in \overline{1, p}$ exist such that $m(t_k + 0) > 0$ and $m(t_k + 0) > m(t)$ for $t \in [0, T]$. Then the following inequalities hold:

$$0 < m(t_k + 0) \leq (1 - L_k)m(t_k)$$

or

$$m(t_k) > 0$$

Hence

$$0 < m(t_k + 0) - m(t_k) \leq -L_k m(t_k) < 0$$

The contradiction obtained shows that in this case the assumption is not true.

This completes the proof of Lemma 1.

Theorem 1. Let the following conditions hold:

1. The functions $v_0, w_0 \in \Omega$ are respectively, lower and upper solutions of problem (1).

2. The function $f: [0, T] \times R \to R$ is continuous and for $v_0(t) \leq v \leq u \leq w_0(t)$ the following inequality holds:

$$f(t, u) - f(t, v) \geq -M(u - v), \qquad t \in [0, T]$$

where $M = \text{const} > 0$.

3. The functions $I_i: R \to R$ for $v_0(t) \leq v \leq u \leq w_0(t)$ satisfy the condition

$$I_i(u) - I_i(v) \geq -L_i(u - v), \qquad i = \overline{1, p}$$

where $0 < L_i < 1$.

Then there exist sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ that are uniformly convergent in the interval $[0, T]$, and their limits $v(t) = \lim_{k\to\infty} v^{(k)}(t)$ and $w(t) = \lim_{k\to\infty} w^{(k)}(t)$ are, respectively, minimal and maximal solutions of the periodic problem (1) in $D(v_0, w_0)$.

Proof. For any function $\eta \in D$ consider the periodic problem

$$\begin{aligned} \dot{u} &= -Mu + f(t, \eta) + M\eta \qquad \text{for} \quad t \neq t_i \\ \Delta u\big|_{t=t_i} &= -L_i u(t_i) + I_i(\eta(t_i)) + L_i \eta(t_i) \\ u(0) &= u(T) \end{aligned} \tag{5}$$

The linear periodic problem (5) has a solution that is unique.

Define the mapping A by the formula $A_\eta = u$, where $u(t)$ is the unique solution of problem (5). It is easily verified that the operator A satisfies the following conditions:

(a) $v_0 \leq Av_0$ and $w_0 \geq Aw_0$.

(b) The operator A is monotone increasing in $D(v_0, w_0)$, i.e. for $\eta_1, \eta_2 \in D(v_0, w_0)$, $\eta_1 \leq \eta_2$, the inequality $A\eta_1 \leq A\eta_2$ holds.

Define the sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ by the equalities

$$v^{(0)} = v_0, \qquad w^{(0)} = w_0, \qquad v^{(k+1)} = Av^{(k)}, \qquad w^{(k+1)} = Aw^{(k)}, \qquad k \geq 0,$$

and obtain that for $t \in [0, T]$ the following inequalities hold:

$$v^{(0)} \leq v^{(1)}(t) \leq \cdots \leq v^{(k)}(t) \leq \cdots \leq w^{(k)}(t) \leq \cdots \leq w^{(1)}(t) \leq w^{(0)}(t)$$

From the last inequalities it follows that the sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ are uniformly convergent in $[0, T]$ and their limits $v(t) = \lim_{k\to\infty} v^{(k)}(t)$ and $w(t) = \lim_{k\to\infty} w^{(k)}(t)$ are solutions of the periodic problem (1).

Let $u \in D(v_0, w_0)$ be an arbitrary solution of (1). We shall show that the inequality $v(t) \leq u(t) \leq w(t)$ holds for $t \in [0, T]$. In fact, for some $k \geq 0$ let the inequalities $v^{(k)}(t) \leq u(t) \leq w^{(k)}(t)$ hold for $t \in [0, T]$. Set $p(t) = v^{(k+1)}(t) - u(t)$. The function $p(t)$ satisfies the following impulsive differential inequality:

$$\begin{aligned} \dot{p}(t) &= f(t, v^{(k)}(t)) - M(v^{(k+1)}(t) - v^{(k)}(t)) - f(t, u) \\ &\leq M(u - v^{(k)}) - M(v^{(k+1)} - v^{(k)}) = Mp(t) \qquad \text{for} \quad t \neq t_i \\ \Delta \rho|_{t=t_i} &\leq L_i(u(t_i) - v^{(k)}(t_i)) - L_i(v^{(k+1)}(t_i) - v^{(k)}(t_i)) \\ &= -L_i p(t_i) \end{aligned}$$

with a periodic condition

$$p(0) = p(T).$$

By Lemma 1 the inequality $p(t) \leq 0$ holds; hence

$$v^{(k+1)} \leq u(t) \qquad \text{for} \quad t \in [0, T].$$

It is obtained in an analogous way that $u(t) \leq w^{(k+1)}(t)$ for $t \in [0, T]$.

By induction it is proved that for $l \geq k$ the inequalities $v^l(t) \leq u(t) \leq w^{(l)}(t)$ hold for $t \in [0, T]$, which shows that $v(t)$ and $w(t)$ are a minimal and maximal solutions of problem (1) in $D(v_0, w_0)$.

This completes the proof of Theorem 1.

Remark 1. If problem (1) has a unique solution $u(t) \in D(v_0, w_0)$ and the conditions of Theorem 1 hold, then by Theorem 1 there exist two sequences of functions tending uniformly from the left and from the right to this solution.

Case II. Let $n \geq 2$.

Theorem 2. Let the following conditions be fulfilled:

1. The functions $v_0(t)$ and $w_0(t)$ are a couple of lower and upper quasisolutions of periodic problem (2) and $v_0(t) \leq w_0(t)$ for $t \in [0, T]$.

2. The function $f:[0, T]\times R^n \to R^n$ $(n \geq 2)$ is continuous and mixed quasimonotone in $[0, T]\times D(v_0, w_0)$.

3. The functions $I_i: R^n \to R^n$ $(n \geq 2)$ are mixed quasimonotone.

4. For $u, v \in R^n$ such that $v_0(t) \leq u \leq v \leq w_0(t)$ for $t \in [0, T]$, the inequalities

$$f_j(t, v_j, [u]_{p_j}, [u]_{q_j}) - f_j(t, u_j, [u]_{p_j}, [u]_{q_j}) \geq -M_j(v_j - u_j)$$
$$I_{ij}(v_j, [u]_{p_j}, [u]_{q_j}) - I_{ij}(u_j, [u]_{p_j}, [u]_{q_j}) \geq -L_{ij}(v_j - u_j)$$

hold, where $M_j \geq 0$, $0 < L_{ij} < 1$, $i = 1, p$, $j = 1, n$.

Then the following assertions are valid:

1. There exist monotone sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ that are uniformly convergent in $[0, T]$ and their limits $v(t) = \lim_{k\to\infty} v^{(k)}(t)$ and $w(t) = \lim_{k\to\infty} w^{(k)}(t)$ are a couple of minimal and maximal quasisolutions of problem (2) in $D(v_0, w_0)$.

2. If the function $u \in D(v_0, w_0)$ is a solution of problem (2), then for $t \in [0, T]$ the following double inequality holds: $v(t) \leq u(t) \leq w(t)$.

Proof. Proof of assertion 1. For any couple of functions $\eta, \mu \in D(v_0, w_0)$ consider the periodic problem

$$\begin{aligned} \dot{u}_j &= -M_j u_j + f_j(t, \eta_j, [\eta]_{p_j}, [\mu]_{q_j}) + M_j \eta_j \qquad \text{for} \quad t \neq t_i \\ \Delta u_j\big|_{t=t_i} &= -L_{ij} u_j(t_i) + I_{ij}(\eta_j(t_i), [\eta(t_i)]_{p_j}, [\mu(t_i)]_{q_j}) + L_{ij}\eta_j(t_i) \\ u_j(0) &= u_j(T), \qquad j = \overline{1, n} \end{aligned} \tag{6}$$

Problem (6) has a unique solution for any fixed couple of functions $\eta, \mu \in D(v_0, w_0)$, which is represented by the formula

$$\begin{aligned} u_j(t) = u_j(0)\, e^{-M_j t} &+ \int_0^T \{f_j(s, \eta_j, [\eta]_{p_j}, [\mu]_{q_j}) \\ &+ M_j \eta_j(s)\}\, e^{M_j(s-t)}\, ds \\ &+ \sum_{0<t_i<t} e^{M_j(t_i - t)}\{L_{ij}\eta_j(t_i) + I_{ij}(\eta_j(t_i), [\eta(t_i)]_{p_j}, [\mu(t_i)]_{p_j})\} \end{aligned} \tag{7}$$

where

$$u_j(0) = (e^{M_j T} - 1)^{-1}\left\{\int_0^T [f_j(s, \eta_j(s), [\eta(s)]_{p_j}, [\mu(s)]_{q_j}) + M_j\eta_j(s)]\, e^{M_j s}\, ds \right.$$
$$\left. + \sum_{0<t_i<t} e^{M_j t_i}[I_{ij}(\eta_j(t_i), [\eta(t_i)]_{p_j}, [\mu(t_i)]_{q_j}) + L_{ij}\eta_j(t_i)]\right\}$$

Define the mapping A: $D(v_0, w_0)\times D(v_0, w_0) \to R^n$ by means of the equality $A(\eta, \mu) = u$, where $u = (u_1, u_2, \ldots, u_n)$ and $u_j(t)$ is the unique solution of problem (6) for the couple of functions η and μ what is given by formula (7).

We shall prove that $v_0 \leq A(v_0, w_0)$. Introduce the notation $v^{(1)} = A(v_0, w_0)$, where $v^{(1)} = (v_1^{(1)}, v_2^{(1)}, \ldots, v_n^{(1)})$ is the unique solution of problem (6) for $\eta = u_0$ and $\mu = w_0$.

Set $p(t) = v_0(t) - v^{(1)}(t)$. Then the following inequalities hold:

$$\begin{aligned} \dot{p}_j(t) &\leq -M_j p_j(t) \qquad \text{for} \quad t \neq t_i \\ \Delta p_j\big|_{t=t_i} &\leq -L_{ij} p_j(t_i) \\ p_j(0) &\leq p_j(T) \end{aligned}$$

By Lemma 1, $p_j(t) \leq 0$, $t \in [0, T]$, or $v_0 \leq A(v_0, w_0)$.

It is proved in an analogous way that $w_0 \geq A(w_0, v_0)$.

Let $\eta, \mu \in D(v_0, w_0)$ be such that $\eta \leq \mu$. Set $u^{(1)} = A(\eta, \mu)$ and $u^{(2)} = A(\mu, n)$. Introduce the notation $p_j(t) = u_j^{(1)}(t) - u_j^{(2)}(t), j \in \overline{1, n}$. From conditions 1, 2, and 4 of Theorem 2 we obtain the following impulsive differential inequalities

$$\begin{aligned} \dot{p}_j(t) &\leq M_j(\mu_j - \eta_j) - M_j(u_j^{(1)} - u_j^{(2)}) + M_j(\eta_j - \mu_j) \\ &= -M_j p_j(t), \qquad \text{for} \quad t \neq t_i \\ \Delta p_j\big|_{t=t_i} &\leq -L_{ij} p_j(t_i) \end{aligned} \tag{8}$$

with a periodic condition

$$p_j(0) = p_j(T), \qquad j = \overline{1, n}$$

By Lemma 1, the inequality $p_j(0) \leq 0, j = \overline{1, n}$, holds. Hence, for $\eta, \mu \in D(v_0, w_0)$, $\eta \leq \mu$, the inequality $A(\eta, \mu) \leq A(\mu, \eta)$ holds.

Define the sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ by the equalities $v^{(0)} = v_0$, $w^{(0)} = w_0$, $v^{(k+1)} = A(v^{(k)}, w^{(k)})$, $w^{(k+1)} = A(w^{(k)}, v^{(k)})$, $k \geq 0$.

Then, in view of what was proved above, the inequalities

$$v^{(0)}(t) \leq v^{(1)}(t) \leq \cdots \leq v^{(k)}(t) \leq \cdots \leq w^{(k)}(t) \leq \cdots \leq w^{(0)}(t)$$

hold for $t \in [0, T]$.

The above inequalities show that the sequences $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ are uniformly convergent in $[0, T]$, and their limits $v(t) = \lim_{k\to\infty} v^{(k)}(t)$ and $w(t) = \lim_{k\to\infty} w^{(k)}(t)$ are a couple of quasisolutions of problem (2) in $D(v_0, w_0)$.

We shall show that (v, w) are a couple of maximal and minimal quasisolutions of (2) in $D(v_0, w_0)$. Let $u_1, u_2 \in D(v_0, w_0)$ be a couple of quasisolutions of problem (2). Then there exists a positive integer k such that

$$v^{(k-1)}(t) \leq u_1(t) \leq w^{(k-1)}(t), \qquad v^{(k-1)}(t) \leq u_2(t) \leq w^{(k-1)}(t)$$

for $t \in [0, T]$. We set $p(t) = v^{(k)}(t) - u_1(t)$. The function $p(t)$ satisfies inequalities (8) and by Lemma 1, $p_j(t) \leq 0$, $j = \overline{1, n}$, i.e., $v_j^{(k)}(t) \leq u_{1j}(t)$ for $t \in [0, T]$, $j = \overline{1, n}$.

It is analogously proved that $v^{(k)}(t) \leq u_2(t)$ and $w^{(k)}(t) \geq u_i(t) (i=1, 2)$ for $t \in [0, T]$.

By induction we obtain that for any positive integer k the inequalities $v^{(k)}(t) \leq u_1(t) \leq w^{(k)}(t)$ and $v^{(k)} \leq u_2(t) \leq w^{(k)}(t)$ hold for $t \in [0, T]$.

Hence the functions v and w are a couple of minimal and maximal quasisolutions of problem (2) in $D(v_0, w_0)$.

Proof of assertion 2. Let $u \in D(v_0, w_0)$ be a solution of problem (2). Then the couple (u, u) can be considered as a couple of quasisolutions of problem (2) in $D(v_0, w_0)$. By assertion 1 of Theorem 2, the inequality $v(t) \leq u(t) \leq w(t)$ holds for $t \in [0, T]$.

This completes the proof of Theorem 2. ■

Example 1. Consider the periodic problem

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \qquad \text{for} \quad t \neq t_1 \\ \dot{x}_2 &= x_1 - 2x_2 \\ \Delta x_i \big|_{t=t_1} &= x_i \qquad i = 1, 2 \\ x_i(0) &= x_i(T) \end{aligned} \tag{9}$$

where $0 < t_1 < T$.

Define the functions $v, w: [0, T] \to R^2$ by the equalities $v = (v_1, v_2)$, $w = (w_1, w_2)$, $v_i(t) \equiv 0$, $(i = 1, 2)$,

$$w_1(t) = \begin{cases} e^{(t_1 - t)}(e^T - 1) & \text{for} \quad t \in [0, t_1] \\ e^{(T + t_1 - t)}(e^T - 1) & \text{for} \quad t \in (t_1, T] \end{cases}$$

$$w_2(t) = \begin{cases} e^{2(t_1 - t)}(e^{2T} - 1) & \text{for} \quad t \in [0, t_1] \\ e^{2(T + t_1 - t)}(e^{2T} - 1) & \text{for} \quad t \in (t_1, T] \end{cases}$$

Choose the numbers $p_j = 0$, $q_j = 1$, $j = 1, 2$, and introduce the notations

$$(x_j, [x]_{p_j}, [y]_{q_j}) = \begin{cases} (x_1, y_2) & \text{for} \quad j = 1 \\ (y_1, x_2) & \text{for} \quad j = 2 \end{cases}$$

It is easily verified that the couple of functions (w, v) are a couple of lower and upper quasisolutions of problem (9) and $v_i(t) \leq w_i(t)$ for $t \in [0, T]$, $i = 1, 2$.

A straightforward verification shows that the conditions of Theorem 2 are satisfied for $M_1 = 1$, $M_2 = 2$, $L_{11} = L_{12} = \frac{1}{2}$.

By Theorem 2 we can construct the couple of minimal and maximal quasisolutions of problem (9) as limits of sequences of functions $\{v^{(k)}(t)\}_0^\infty$ and $\{w^{(k)}(t)\}_0^\infty$ where $v_i^{(0)}(t) = 0$, $w_i^{(0)}(t) = w_i(t)$, $i = 1, 2$.

For functions $v^{(1)}(t)$ and $w^{(1)}(t)$, by formula (7) we obtain

$$v_1^{(1)}(t)=\begin{cases}(e^{2T}-1)\,e^{2t_1-t}(e^{T-t_1}+e^{-t_1}-e^{-t}) & \text{for}\quad t\in[0,t_1]\\ (e^{2T}-1)\,e^{2t_1-t+T}(e^{T-t_1}+e^{-t_1}-e^{T-t}) & \text{for}\quad t\in(t_1,T]\end{cases}$$

$$v_2^{(1)}(t)=\begin{cases}(e^{T}+1)^{-1}(e^{T}-1)\,e^{T+t_1-t} & \text{for}\quad t\in[0,t_1]\\ (e^{T}+1)^{-1}(e^{T}-1)\,e^{T+t_1-t}(1+e^{T}-e^{T-t}) & \text{for}\quad t\in(t_1,T]\end{cases}$$

$$w_1^{(1)}(t)=\begin{cases}e^{t_1-t}(e^{T}-\frac{1}{2}) & \text{for}\quad t\in[0,t_1]\\ e^{T+t_1-t}(e^{2T}-\frac{1}{2}) & \text{for}\quad t\in(t_1,T]\end{cases}$$

$$w_2^{(1)}(t)=\begin{cases}e^{2(t_1-t)}(e^{2T}-\frac{1}{2}) & \text{for}\quad t\in[0,t_1]\\ e^{2(T+t_1-t)}(e^{2T}-\frac{1}{2}) & \text{for}\quad t\in(t_1,T]\end{cases}$$

By Theorem 2 for $t\in[0,T]$ the inequalities

$$v_i^{(1)}(t)\le u_i(t)\le w_i^{(1)}(t)\qquad (i=1,2)$$

hold, where $(u_1(t),u_2(t))$ is a solution of problem (9).

ACKNOWLEDGEMENT

This work was supported financially by the Committee for Science of the Council of Ministers, Bulgaria, grant 61.

REFERENCES

Bernfeld, S. R., and Lakshmikantham, V. (1982). Linear monotone method for nonlinear boundary value problems, *Rocky Mountain Journal*, **12**, 807–815.

Deimling, K., and Lakshmikantham, V. (1980). Quasisolutions and their role in the qualitative theory of differential equations, *Nonlinear Analysis*, **4**, 657–663.

Du, S. W. and Lakshmikantham, V. (1982). Monotone iterative technique for differential equations in Banach space, *Journal of Mathematical Analysis and Applications*, **87**, 454–459.

Hristova, S. G., and Bainov, D. D. (1986). Periodic solutions of quasilinear non-autonomous systems with impulses, *Mathematical Methods in the Applied Sciences*, **8**, 247–255.

Lakshmikantham, V., and Leela, S. (1984). Remarks on first and second order periodic boundary value problems, *Nonlinear Analysis*, **8**, 281–287.

Lakshmikantham, V., and Vatsala, A. S. (1981). Quasi-solutions and monotone method for systems of nonlinear boundary value problems, *Journal of Mathematical Analysis and Applications*, **79**, 38–47.

Lakshmikantham, V., Leela, S., and Oguztorelli, M. N. (1981). Quasisolutions, vector Lyapunov functions and monotone method, *IEEE Transactions on Automatic Control* **26**.

Mil'man, V. D., and Myshkis, A. D. (1960). On the stability of motion in the presence of impulses, *Sibirski Math. Zhurnal*, **1**, 233–237 (in Russian).

Perestyuk, N. A., and Shovkoplyas, V. N. (1973). Periodic solutions of nonlinear differential equations with impulse effect, *Ukr. Math. Zhurnal* **173**(5), 517–524 (in Russian).

Samoilenko, A. M., and Perestyuk, N. A. (1982). Periodic and almost periodic solutions of differential equations with impulse effect, *Ukr. Math. Zhurnal*, **1982**(1), 66–73 (in Russian).

Vatsala, A. S. (1983). On the existence of periodic quasi-solutions for first order systems, *Nonlinear Analysis*, **7**, 1283–1289.